JOURNAL OF APPROXIMATION THEORY 60, 174–182 (1990)

Approximation of Continuous Functions by T-Means of Fourier Series

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Communicated by R. Bojanic

Received March 25, 1988

The degree of approximation of a function $f \in C_{2\pi}$ by the T-means of its Fourier series is examined. The corresponding result for the conjugate function $\tilde{f}$ is also obtained. It is shown that our results are the best possible for the class H^{ω}. A number of interesting special cases are studied. © 1990 Academic Press, Inc.

Let $T = (t_{n,k})_{k,n=0}^{\infty}$ be a summability matrix with the properties

$$t_{n,k} \geqslant t_{n,k+1}, \qquad k = 0, 1, ..., n = 0, 1, ...; \tag{1}$$

$$\sum_{k=0}^{\infty} t_{n,k} = 1, \qquad n = 0, 1, 2, ...; \tag{2}$$

$$\lim_{n \to \infty} t_{n,0} = 0. \tag{3}$$

We shall establish the exact order of approximation of the T-means of Fourier series. Note that (2) is just a normalization and (3) is needed for the regularity of the matrix T. The crucial property in our investigations will be the decreasing character of the rows of T, i.e., the property (1).

* The research work of the first author was supported by Kuwait University Research Council Grant SM038, while that of the second author was partially supported by the Hungarian National Foundation for Scientific Research under Grant 1157.

0021-9045/90 $3.00

Let $f \in C_{2\pi}$ have Fourier series $f(x) \sim a_0/2 + \sum_{v=1}^{\infty} (a_v \cos vx + b_v \sin vx)$. T-means of Fourier series of f are defined as

$$T_n(f, x) = \sum_{k=0}^{\infty} t_{n,k} s_k(f, x), \tag{4}$$

where $s_k(f, x)$ denotes the kth partial sum of the Fourier series of f. Furthermore let

$$\sigma_n(f, x) = \frac{1}{n+1} \sum_{n=0}^{n} s_k(f, x)$$

be the Fejér sum of f. Since

$$\begin{aligned} T_n(f, x) &= \lim_{m \to \infty} \sum_{k=0}^{m} t_{n,k} s_k(f, x) \\ &= \lim_{m \to \infty} \left\{ \sum_{k=0}^{m-1} (t_{n,k} - t_{n,k+1})(k+1)\, \sigma_k(f, x) + t_{n,m}(m+1)\, \sigma_n(f, x) \right\}, \end{aligned} \tag{5}$$

$\sigma_k(f, x) \to f(x)$ uniformly as $k \to \infty$ and by (1) and (2), $(m+1) t_{n,m} \to 0$ as $m \to \infty$, we conclude that (4) is a correct definition, and by (3), $T_n f \to f$ uniformly as $n \to \infty$.

Since σ_k's are positive operators, the above computation shows that T_n's are positive convolution operators and hence (see [1, Theorem 2.4]) they are saturated with the saturation order $\{t_{n,0}\}$ and we have actually

$$T_n f - f = O(\omega(f, \sqrt{t_{n,0}})), \tag{6}$$

where $\omega(f, \delta)$ denotes the modulus of continuity of f. However, as will be seen below, this is a rather weak estimate, and much better ones can be obtained if we look upon T_n as a summability method.

THEOREM 1. *Suppose T satisfies* (1)–(3). *Then*

$$|T_n f - f| \leqslant C_1 \sum_{k=0}^{\infty} t_{n,k}\, \omega\left(f, \frac{1}{k+1}\right) \tag{7}$$

and

$$|T_n \tilde{f} - \tilde{f}| \leqslant C_1 \left\{ t_{n,0}\, \omega^*(1) + \sum_{k=1}^{\infty} t_{n,k} \left((k+1)\, \omega^*\left(\frac{1}{k+1}\right) - k\omega^*\left(\frac{1}{k}\right) \right) \right\}, \tag{8}$$

where

$$\omega^*(t) = \int_0^t \frac{\omega(u)}{u}\, du,$$

and ω *is an arbitrary majorant of* $\omega(f; \cdot)$.

Let ω be a modulus of continuity and $H^\omega = \{f/\omega(f, t) = O(\omega(t))\}$. The necessary and sufficient condition that for every $f \in H^\omega$ its trigonometric conjugate function $\tilde{f}$ be bounded is that

$$\int_0^1 \frac{\omega(u)}{u}\, du < \infty. \tag{9}$$

Now we show that in the class H^ω the estimates given by Theorem 1 are in general the best possible.

THEOREM 2. *Let* ω *be an arbitrary modulus of continuity. Then there are functions* $f_0, f_1 \in H^\omega$ *such that*

$$|T_n f_0(0) - f_0(0)| \geqslant \sum_{k=0}^{\infty} t_{n,k}\, \omega\left(\frac{1}{k+1}\right) \tag{10}$$

and

$$\begin{aligned} |T_n \tilde{f}_1(0) - \tilde{f}_1(0)| \geqslant\ & t_{n,0}\, \omega^*(1) \\ & + \sum_{k=1}^{\infty} t_{n,k} \left((k+1)\, \omega^*\left(\frac{1}{k+1}\right) - k\omega^*\left(\frac{1}{k}\right)\right). \end{aligned} \tag{11}$$

As a consequence of Theorem 1 we get

COROLLARY. *Under the assumptions of Theorem* 1 *we have*

$$T_n f - f = O\left(\omega\left(f, t_{n,0} \log \frac{1}{t_{n,0}}\right)\right) \tag{12}$$

and

$$T_n \tilde{f} - \tilde{f} = O(\omega^*(f, t_{n,0})). \tag{13}$$

Since for large n, $t_{n,0} \log(1/t_{n,0})$ is much smaller than $\sqrt{t_{n,0}}$, (12) is a better estimate than (6). Furthermore for $T_n = \sigma_n$ and $f \in \operatorname{lip} 1$, (12) yields

$$\sigma_n f - f = O\left(\frac{\log n}{n}\right) \tag{14}$$

and since in this case ($t_{n,0}=1/(n+1)$) for the Lip 1 function

$$f(x)=\sum_{n=1}^{\infty}\frac{\cos kx}{k^2}$$

we have $|\sigma_n f(0)-f(0)|\geqslant \frac{1}{2}((\log n)/n)$, the logarithmic term can be neither omitted in (12) nor can it be replaced by a smaller quantity. Note, however, that in (13) there is no logarithmic factor. Finally, we remark that (13) is the sharpest possible estimate in most cases.

Remarks. 1. From the proof below it will follow that the better estimate

$$|T_n f-f|\leqslant C_0\sum_{k=0}^{\infty} t_{n,k}E_k(f),$$

where $E_k(f)$ denotes the best approximation of f by trigonometric polynomials of order at most k, is still true.

2. Summability methods with a continuous parameter can be similarly treated.

3. The right-hand side of (8) obviously can be increased to have the estimate

$$|T_n\tilde{f}-\tilde{f}|\leqslant C\sum_{k=0}^{\infty} t_{n,k}\omega^*\left(\frac{1}{k+1}\right)$$

but this is a considerably weaker estimate than (8), e.g., (8) yields that $f\in \mathrm{Lip}\,1$ implies

$$|T_n\tilde{f}-\tilde{f}|=O(t_{n,0}),$$

in particular $\sigma_n\tilde{f}-\tilde{f}=O(n^{-1})$.

After the proofs we will give concrete applications of Theorems 1 and 2.

Proof of Theorem 1. It is well known (see [2]) that if $E_k(f)$ denotes the best approximation of f by trigonometric polynomials of order at most k, then

$$\frac{1}{k}\sum_{v=k+1}^{2k}|s_v(f)-f|\leqslant CE_k(f),$$

hence

$$|\sigma_n f-f|\leqslant\frac{C}{n+1}\left(E_0(f)+\sum_{2^j\leqslant n}2^jE_{2^j}(f)\right)$$
$$\leqslant\frac{C}{n+1}\sum_{v=0}^{n}E_v(f).$$

By computation similar to that in (5) we have

$$\begin{aligned}
|T_n f - f| &= \left| \sum_{k=0}^{\infty} (t_{n,k} - t_{n,k+1}) \sum_{v=0}^{k} (s_v(f) - f) \right| \\
&\leqslant \sum_{k=0}^{\infty} (t_{n,k} - t_{n,k+1}) C \sum_{v=0}^{k} E_v(f) \\
&= C \sum_{k=0}^{\infty} t_{n,k} E_k(f).
\end{aligned}$$

This and Jackson's theorem yield (7). Equation (8) follows similarly if we note that by [5, Theorem 2]

$$|\sigma_n \tilde{f} - \tilde{f}| \leqslant C \int_0^{1/(n+1)} \frac{\omega(u)}{u}\, du \leqslant C\omega^* \left(\frac{1}{n+1}\right), \qquad n = 0, 1, 2, \ldots.$$

Proof of Theorem 2. Without loss of generality we may assume that ω is concave, since for every modulus of continuity ω there is a concave modulus of continuity $\bar{\omega}$ with the property

$$\omega(\delta) \leqslant \bar{\omega}(\delta) \leqslant 2(\omega(\delta)) \qquad \text{(see [3, p. 45])}.$$

Consider the function

$$f_0(x) = \sum_{v=1}^{\infty} \left(\omega\left(\frac{1}{v}\right) - \omega\left(\frac{1}{v+1}\right)\right) \cos vx.$$

It was proved in [4, Lemma 4] that $f_0 \in H^\omega$. But $f_0(0) - s_k(f_0, 0) = \sum_{v=k+1}^{\infty} (\omega(1/v) - \omega(1/(v+1))) = \omega(1/(k+1))$ and thus (10) is true.

In a similar manner, since

$$\tilde{f}(0) - T_n \tilde{f}(0) = \sum_{k=0}^{\infty} (t_{n,k} - t_{n,k+1})(k+1)(\tilde{f}(0) - \sigma_k \tilde{f}(0)),$$

to prove (11) it is enough to verify that for some function $f_1 \in H^\omega$

$$\tilde{f}_1(0) - \sigma_k \tilde{f}_1(0) \geqslant C \int_0^{1/(k+1)} \frac{\omega(u)}{u}\, du, \qquad k = 0\ 1, 2, \ldots. \tag{15}$$

If $\omega(\delta) \leqslant k\delta$, then let $f_1(x) = \sin x$. For this $\tilde{f}(0) - \sigma_n \tilde{f}(0) \geqslant 1/(n+1)$ can be readily seen.

If $\omega(\delta) \neq O(\delta)$, then by the concavity of ω, actually $\lim_{\delta \to 0+} (\omega(\delta)/\delta) = \infty$

holds. In this case it was shown in [5, Theorem 2] that with appropriate constant B the function

$$\begin{aligned} f_1(x) = \sin x - \sum_{k=2}^{\infty} \left(\omega\left(\frac{1}{k}\right) - \frac{k-1}{k}\, \omega\left(\frac{1}{k-1}\right)\right) \sin kx \\ - B \sum_{k=2}^{\infty} \left(2k\omega\left(\frac{1}{k}\right) - (k-1)\omega\left(\frac{1}{k-1}\right) \right. \\ \left. - (k+1)\omega\left(\frac{1}{k+1}\right)\right) \sin kx \end{aligned}$$

belongs to the class H^{ω} and satisfies (15).

Proof of the Corollary. Applying the inequality

$$\omega\left(\frac{1}{k+1}\right) \leqslant \left(\left(\frac{M}{k+1}\right) + 1\right) \omega\left(\frac{1}{M}\right), \qquad k \leqslant M \tag{16}$$

in (7) we have

$$\begin{aligned} |T_n f - f| &\leqslant C \left(\sum_{k=0}^{M} + \sum_{k=M+1}^{\infty} \right) t_{n,k}\, \omega\left(f, \frac{1}{k+1}\right) \\ &\leqslant C \sum_{k=0}^{M} t_{n,k} \left(\frac{M}{k+1} + 1\right) \omega\left(f, \frac{1}{M}\right) \\ &\quad + C\omega\left(f, \frac{1}{M}\right) \\ &\leqslant C\left(1 + t_{n,0} M \sum_{k=0}^{M} \frac{1}{k+1}\right) \omega\left(f, \frac{1}{M}\right) \\ &\leqslant C(1 + t_{n,0} M \log M)\, \omega\left(f, \frac{1}{M}\right). \end{aligned}$$

Setting

$$M = \left[\frac{1}{t_{n,0} \log(1/t_{n,0})} \right]$$

we get (12).

Now coming to the proof of (13) we know there is a concave modulus of continuity $\bar{\omega}$ such that

$$\omega(f, \delta) \leqslant \bar{\omega}(\delta) \leqslant 2\omega(f, \delta)$$

hence the analogue of (16) holds in the form

$$\omega^*\left(f, \frac{1}{k}\right) \leqslant \bar{\omega}^*\left(\frac{1}{k}\right) \leqslant \frac{M}{k}\,\bar{\omega}^*\left(\frac{1}{M}\right)$$
$$\leqslant \frac{2M}{k}\,\omega^*\left(f, \frac{1}{M}\right), \qquad k \leqslant M$$

(note that together with $\bar{\omega}$, $\bar{\omega}^*$ is also concave) and hence (8) yields

$$|T_n \bar{f} - \bar{f}| \leqslant C\left(t_{n,0}\omega^*(1) + \sum_{k=1}^{\infty} (t_{n,k-1} - t_{n,k})\, k\omega^*\left(\frac{1}{k}\right)\right)$$
$$\leqslant C\left(t_{n,0} 2M\omega^*\left(\frac{1}{M}\right) + \sum_{k=1}^{M} (t_{n,k-1} - t_{n,k})\, \frac{2M}{k}\, k\omega^*\left(\frac{1}{M}\right)\right.$$
$$\left. + \sum_{k=M+1}^{\infty} (t_{n,k-1} - t_{n,k})\, k\omega^*\left(\frac{1}{M}\right)\right)$$
$$\leqslant C(Mt_{n,0} + Mt_{n,0} + Mt_{n,M} + 1)\,\omega^*\left(\frac{1}{M}\right)$$

and for $M = [1/t_{n,0}]$ we get (13).

Examples

Below we give examples for Theorems 1 and 2. To make our discussion shorter we write

$$T_n(H^\omega) \asymp \gamma_n$$

to indicate that for each $f \in H^\omega$ we have

$$|T_n f - f| = O(\gamma_n), \qquad n = 1, 2, \ldots$$

and there are an $f \in H^\omega$ and x_0 such that

$$|T_n f(x_0) - f(x_0)| \geqslant C\gamma_n, \qquad n = 1, 2, \ldots.$$

Some of the "asymptotics" below are well known, we only mention them because they are immediate consequences of Theorems 1 and 2.

1. (C, α) *means*. Let $C_n^\alpha = \{A_{n-k}^{\alpha-1}/A_n^\alpha \mid 0 \leqslant k \leqslant n,\ n = 0, 1, \ldots\}$.[1] These

[1] A_k^α are the binomial coefficients.

satisfy (1)–(3) for $\alpha \geqslant 1$ and from Theorems 1 and 2 we get by simple computation

$$C_n^\alpha(H^\omega) \asymp \frac{1}{n} \int_{1/n}^{\pi} \frac{\omega(u)}{u^2}\, du.$$

2. An example with continuous parameter is that of *Abel means*. Let $A_r = \{(1-r)\, r^k,\ 0<r<1,\ k=0, 1, \dots\}$. For this we have

$$A_r(H^\omega) \sim (1-r) \int_{1-r}^{\pi} \frac{\omega(u)}{u^2}\, du, \qquad r \to 1-0.$$

3. *Riesz means.* Let $\lambda = \lambda(n)$ be a concave sequence tending to ∞ and let

$$R^\lambda = \left\{ \frac{\lambda(k+1) - \lambda(k)}{\lambda(n)},\ k=0, \dots, n-1,\ n=1, 2, \dots \right\}.$$

Here we may assume that the function $\lambda(x)$ is defined for all $x \geqslant 0$ and that it is concave. Then

$$R_n^\lambda(H^\omega) \asymp \frac{1}{\lambda(n)} \int_1^n \lambda'(t)\, \omega\left(\frac{1}{t}\right) dt.$$

4. For convex λ, the *Nörlund means*

$$N^\lambda = \left\{ \frac{\lambda(n-k) - \lambda(n-k-1)}{\lambda(n)},\ k=0, \dots, n-1,\ n=1, 2, \dots \right\}$$

satisfy

$$N_n^\lambda(H^\omega) \asymp \frac{1}{\lambda_n} \int_1^n \lambda'(n-t)\, \omega\left(\frac{1}{t}\right) dt.$$

5. The *L-method* is defined by

$$L_r f(x) = \frac{1}{\log(1/(1-r))} \sum_{k=1}^{\infty} \frac{r^k}{k}\, s_k(f, x).$$

This method does not satisfy condition (1) but it "almost" satisfies it, i.e., it satisfies with

$$t_{r,k} = \frac{r^k}{k \log(1/(1-r))}$$

the inequality $t_{r,k} \geqslant t_{r,k+1}$, $k = 1, 2, \ldots$. (Note that in (1) we would need the same inequality for $k = 0$, as well.)

Theorems 1 and 2 can be applied in such cases as well by a simple modification. Let

$$L_r^* f(x) = \frac{1}{1 + \log(1/(1-r))} \left(L_r f(x) + \frac{s_0(f, x)}{\log(1/(1-r))} \right).$$

Then for L_r^* we have (1)–(3), hence Theorem 1 is true for it,

$$|L_r^* f - f| \leqslant C \frac{1}{\log(1/(1-r))} \left(\omega(1) + \sum_{k=1}^{\infty} \frac{r^k}{k} \omega\left(\frac{1}{k+1}\right) \right), \tag{17}$$

and this obviously implies the same estimate for L_r. Now (17) with L_r instead of L_r^* easily implies that

$$L_r(H^\omega) \asymp \frac{1}{\log(1/(1-r))} \int_{1-r}^{\pi} \frac{\omega(u)}{u} \, du.$$

Note that all the Lip α, $\alpha > 0$, spaces have the optimal order of approximation $\{(\log(1/(1-r)))^{-1}\}$.

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